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ON THE TIME-DEPENDENT LÉVÊQUE PROBLEM*

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Abstract—The classical steady state Lévêque problem is generalized to include the time dependence. Exact analytical solutions are presented for the surface heat flux due to a time step in the surface temperature and for the case of the surface temperature due to a time step in the wall heat flux. The initial and final time behavior of the solution is explored analytically and this is supplemented by a numerical evaluation for the entire time span in the case of a step in the surface temperature.

NOMENCLATURE

Ai,	Airy function;
Ai′,	derivative of the Airy function;
$c_{1,2,\ldots n}$,	zeros of the Airy function Ai;
$c'_{1,2,\ldots n}$,	zeros of Ai′;
$\tilde{c}$,	asymptotic representation of c_n;
$E(\tau^+)$,	error term in equation (28) and given by equation (30);
I_ν,	modified Bessel function of the first kind, order ν;
K_ν,	modified Bessel function of the second kind, order ν;
k,	thermal conductivity of the fluid;
p, s,	Laplace transform variables;
q_0,	prescribed wall heat flux;
q_w,	wall heat flux for the case of prescribed wall temperature;
$q_{w,ss}$,	steady state wall heat flux;
r_n,	residue, given by equation (19);
T,	temperature rise over that at infinity;
T^+,	$= T/T_w$, for the case of a prescribed wall temperature;
	$= Tk\, u_\infty\, \sigma^{1/3}/q_0 v$, for the case of a prescribed wall;
t,	= time;
t^+,	$= t\, u_\infty^2/\nu\sigma^{1/3}$;
u,	fluid velocity component in x-direction;
u_∞,	free stream fluid velocity component in x-direction;
v,	fluid velocity component in y-direction;
x,	distance along the wall;
x^+,	$= x\rho u_\infty^3/\nu\tau_0$;
y,	distance perpendicular to the wall;
y^+,	$= yu_\infty\sigma^{1/3}/\nu$.

Greek symbols

α,	$= 9(3\,\pi/8)^{2/3}$;
Γ,	gamma function;
κ,	thermal diffusivity of the fluid;
ν,	kinematic viscosity of the fluid;
ρ,	density of the fluid;
σ,	Prandtl number, (ν/κ);
τ^+,	$= t^+/(x^+)^{2/3}$;
τ_0,	wall shear stress.

Subscripts

w,	conditions at the wall;
ss,	steady state conditions.

INTRODUCTION

KNOWLEDGE of the characteristics of the unsteady convective heat-transfer process is becoming more and more significant in the design of control systems involving heat exchange

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devices. A detailed review is given in [1] for what has been done in this area of transient forced-convection heat transfer. Only some of the works, dealing with the case of external flow, i.e. flow over a surface which may be of boundary layer type, will be mentioned in the following.

Cess [2] dealt with the case of incompressible laminar boundary layer flow over a flat plate with a step jump in wall temperature. Cess used the series expansion of the Blasius velocity function near the wall. He obtained series solutions for the wall heat flux for short and for long times and these were joined together in the Laplace transform plane by an approximate method. Upon inverting back to the physical plane a solution was obtained valid for all time. Goodman [3] attempted the same problem as in [2] and the wall heat flux was determined by the integral method. Riley [4] treated again the same problem. He used also the series expansion of the Blasius velocity function near the wall and found the solution for small times, in the form of a series in powers of $\tau^{\frac{1}{2}}$ where $\tau = u_{\infty}t/x$. The first two terms agree with those of Cess [1]. For large times, or more precisely, in the final approach to the steady state Riley showed that the departure from the steady state is concentrated near the wall. Therefore the velocity components were again replaced by their values near the wall and it was shown that the steady state is attained in an exponential manner. Chambré [5] treated the problem of slug flow (i.e. uniform velocity distribution) over a flat plate of appreciable thermal capacity which contained time-dependent heat sources. A closed form solution for the transient surface temperature was obtained by double Laplace transformation methods. An analysis was presented by Chao and Jeng [6] for the unsteady, incompressible, laminar forced-convection heat transfer at a two-dimensional and an axisymmetrical (front) stagnation point due to an arbitrarily prescribed wall temperature or heat flux variation. Two asymptotic solutions valid for small and large times, respectively, were found and joined. The key to the small time solution is the change of the energy equation in the Laplace transform plane to an ordinary differential equation with a large parameter which is treated accordingly. For large times the energy equation was integrated and the method of steepest descent was used in the evaluation of the integrals. Chao and Jeng suggested the use of their method to other problems. As examples they show results of their technique when applied to the problem of [2], [3] and [4].

None of the above works, except for [5] (uniform velocity distribution), offers an exact solution uniformly valid at all times. The present analysis treats the time dependent Lévêque problem which is described in the next Section. The simple nature of this problem allows us to carry on an exact analysis. The closed form solution obtained may serve as a reliable reference to check techniques developed for the treatment of more complicated problems.

THE STATEMENT OF THE PROBLEM

The present analysis is based on the assumption of a linear velocity profile. Hence, the results obtained should hold whenever this assumption is realistic, e.g. in both cases of laminar and turbulent flows when the Prandtl number σ is very large or more precisely when $\sigma \to \infty$. The starting point is the laminar form of the energy equation

$$\left.\frac{\partial T}{\partial t} + u(x, y)\frac{\partial T}{\partial x} + v(x, y)\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}; \quad \right\} (1)$$
$$t > 0, \quad x > 0, \quad y > 0.$$

Here the co-ordinates (x, y) are measured along and normal to the wall respectively and (u, v) represent the velocity components in these directions. The viscous dissipation and axial conduction are neglected in equation (1) and the physical properties are assumed constant. The last assumption restricts the analysis to small temperature changes. For a time step in the wall temperature of magnitude T_w the side conditions for $T(t, x, y)$ are

$$\left.\begin{aligned} &T(0, x, y) = 0, \quad x > 0, \quad y > 0; \\ &T(t, 0, y) = 0, \quad t > 0, \quad y > 0; \\ &T(t, x, 0) = T_w, \quad t > 0, \quad x > 0; \\ &T(t, x, \infty) = 0, \quad t > 0, \quad x > 0. \end{aligned}\right\} \quad (2)$$

Consider the linear velocity profile $u = \tau_0 y/\mu$, where the wall shear stress τ_0 is assumed to be constant. Hence from the continuity equation one readily sees that $v = 0$. The neglect of the term $v(\partial T/\partial y)$ results in a considerable simplification of equation (1). Introducing the following dimensionless variables:

$$\left.\begin{aligned} x^+ &= \frac{x\rho u_\infty^3}{\nu\tau_0}, \qquad y^+ = \frac{yu_\infty}{\nu}\sigma^{\frac{1}{3}}, \\ t^+ &= \frac{tu_\infty^2}{\nu\sigma^{\frac{1}{3}}} \quad \text{and} \quad T^+ = \frac{T}{T_w} \end{aligned}\right\} \quad (3)$$

equations (1) and (2) reduce to

$$\left.\begin{aligned} &\frac{\partial T^+}{\partial t^+} + y^+ \frac{\partial T^+}{\partial x^+} = \frac{\partial^2 T}{\partial y^{+2}}, \\ &t^+ > 0, \quad x^+ > 0, \quad y^+ > 0 \end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned} &T^+(0, x^+, y^+) = 0, \quad x^+ > 0, \quad y^+ > 0; \\ &T^+(t^+, 0, y^+) = 0, \quad t^+ > 0, \quad y^+ > 0 \end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned} &T^+(t^+, x^+, 0) = 1, \quad t^+ > 0, \quad x^+ > 0; \\ &T^+(t^+, x^+, \infty) = 0, \quad t^+ > 0, \quad x^+ > 0. \end{aligned}\right\} \quad (6)$$

The purpose of the present analysis is to obtain an exact solution for the wall heat flux based on the system of equations (4) to (6). The determination of the wall temperature in the case of a time step in the wall heat flux is closely related to the above problem and the results for this case are given at the end of the paper.

ANALYSIS

The present problem is solved by application of a double Laplace transformation with respect to the variables t^+ and x^+. Their images are denoted by s and p respectively. Provided the transformation of the temperature function $T^+(t^+, x^+, y^+)$ exists, (y^+ is treated as a parameter), the double Laplace transform of T^+ is defined by

$$\begin{aligned} \varphi(s, p, y^+) &\equiv \mathscr{L}_{x^+}\mathscr{L}_{t^+}\{T^+(t^+, x^+, y^+)\} \\ &= \mathscr{L}_{t^+}\mathscr{L}_{x^+}\{T^+(t^+, x^+, y^+)\} \\ &\equiv \int_0^\infty \int_0^\infty T^+(t^+, x^+, y^+) \exp(-st^+ \\ &\qquad - px^+)\,\mathrm{d}t^+\,\mathrm{d}x^+. \end{aligned} \quad (7)$$

Details of this method are given in [7]. If one applies the integral operator (7) to equation (4) one obtains, by using the side conditions (5), the ordinary differential equation for the transform φ

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}y^{+2}} - (s + py^+)\varphi = 0. \quad (8)$$

Equation (6) when transformed to the (s, p, y^+) domain becomes

$$\varphi(s, p, 0) = 1/sp, \qquad \varphi(s, p, \infty) = 0. \quad (9)$$

Now let

$$\rho = s + py^+ \quad (10)$$

then for $\varphi(s, p, \rho)$ the relations (8) and (9) reduce to

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\rho^2} - \frac{\rho}{p^2}\varphi = 0 \quad (11)$$

$$\varphi(s, p, s) = 1/sp, \qquad \varphi(s, p, \infty) = 0. \quad (12)$$

The differential equation (11) is satisfied by the Airy function [8]. The particular solution to this equation which obeys the condition at infinity is

$$\varphi(s, p, \rho) = C\,\mathrm{Ai}(\rho/p^{\frac{2}{3}}). \quad (13)$$

C is evaluated from the boundary condition at the wall ($y^+ = 0$, i.e. $\rho = s$) with the result that

$$\varphi(s, p, \rho) = \frac{\mathrm{Ai}(\rho/p^{\frac{2}{3}})}{sp\,\mathrm{Ai}(s/p^{\frac{2}{3}})}. \quad (14)$$

In the present problem the main interest lies in the evaluation of the wall heat flux, $q_w = -k(\partial T/\partial y)(t, x, 0)$, which in the transform domain is defined by

$$
\begin{aligned}
&\mathscr{L}_{x^+}\mathscr{L}_{t^+}\{q_w \nu / u_\infty T_w k \sigma^{\frac{1}{3}}\} \\
&= \mathscr{L}_{x^+}\mathscr{L}_{t^+}\left\{-\frac{\partial T^+}{\partial y^+}(t^+, x^+, 0)\right\} \qquad (15)\\
&= -p\frac{\mathrm{d}\varphi}{\mathrm{d}\rho}(s, p, s).
\end{aligned}
$$

From equations (14) and (15) one computes

$$
\begin{aligned}
&-p\frac{\mathrm{d}\varphi}{\mathrm{d}\rho}(s, p, s) \equiv f(s, p) \\
&= -\frac{1}{sp^{\frac{2}{3}}}\frac{\mathrm{Ai}'(s/p^{\frac{2}{3}})}{\mathrm{Ai}(s/p^{\frac{2}{3}})}. \qquad (16)
\end{aligned}
$$

The function $f(s, p)$ is holomorphic in the upper right-hand quadrants of the (s, p)-planes if $\mathscr{R}(s) > 0$ and $\mathscr{R}(p) > 0$, where $\mathscr{R}$ denotes the real part of s or p.

The major task now is to invert equation (16) to the physical (t^+, x^+) domain. According to the properties of the double Laplace transform [7] the order of inversions with respect to s and p can be interchanged. In the following the function $f(s, p)$ will first be inverted with respect to s treating p as a parameter. This inversion is denoted by $\mathscr{L}_s^{-1}$.

The functions Ai and Ai′ are both entire functions of s, [8]. They have a sequence of simple zeros on the negative real axis and no zeros elsewhere. The zeros of Ai′ are different from those of Ai. Hence the function $f(s, p)$ for $\mathscr{R}(p) > 0$ is an analytic function everywhere in the s-plane except for an infinite number of isolated singularities (simple poles) on the negative real axis at $s_n (n = 0, 1, 2, \ldots)$. Note that s_0 is the singularity at $s = 0$ caused by the factor $1/s$ in equation (16) while $s_n = -c_n p^{\frac{2}{3}}$ for $n \geqslant 1$, with $\mathscr{R}(p) > 0$. If $r_n(t^+, p)$ is the residue of $\exp(st^+) f(s, p)$ at $s = s_n$ the inversion of $f(s, p)$ with respect to s may be written as, [9] p. 186.

$$
\begin{aligned}
&\mathscr{L}_s^{-1}\{f(s, p)\} \equiv F(t^+, p) \\
&= \sum_{n=0}^{\infty} r_n(t^+, p), \quad t^+ > 0, \quad \mathscr{R}(p) > 0\,. \qquad (17)
\end{aligned}
$$

To calculate the residues $r_n(t^+, p)$ the function $\exp(st^+) f(s, p)$ is expressed in the form

$$
\exp(st^+) f(s, p) = \frac{\exp(st^+)\, P(s, p)}{Q(s, p)} \qquad (18)
$$

where

$$
P(s, p) = -\mathrm{Ai}'(s/p^{\frac{2}{3}})
$$

and

$$
Q(s, p) = sp^{\frac{2}{3}}\mathrm{Ai}(s/p^{\frac{2}{3}}).
$$

The analytic functions $\exp(st^+)\, P(s, p)$ and $Q(s, p)$ satisfy the conditions $Q(s_n, p) = 0$, $Q'(s_n, p) \neq 0$ and $\exp(s_n t^+)\, P(s_n, p) \neq 0$. The residue of $\exp(st^+) f(s, p)$ at the simple pole s_n has the value

$$
\left.
\begin{aligned}
r_n(t^+, p) &= \frac{\exp(s_n t^+)\, P(s_n, p)}{Q'(s_n, p)}, \\
&\qquad n \geqslant 0, \quad t^+ > 0, \quad \mathscr{R}(p) > 0.
\end{aligned}
\right\} \qquad (19)
$$

Using the equations (17), (18), (19) and the values

$$
\begin{aligned}
\mathrm{Ai}(0) &= 1/[3^{\frac{2}{3}}\,\Gamma(\tfrac{2}{3})], \\
\mathrm{Ai}'(0) &= -1/[3^{\frac{1}{3}}\,\Gamma(\tfrac{1}{3})],
\end{aligned}
$$

one obtains for the first inversion step

$$
\begin{aligned}
F(t^+, p) &= \left(\frac{3^{\frac{1}{3}}\,\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right)\frac{1}{p^{\frac{2}{3}}} \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{c_n p^{\frac{2}{3}}}\exp\left[-(c_n t^+)\, p^{\frac{2}{3}}\right], \\
&\qquad\qquad t^+ > 0, \quad \mathscr{R}(p) > 0. \qquad (20)
\end{aligned}
$$

The c_n are the zeros of the Airy function. The first fifty values of c_n have been calculated by Miller [8].

Turning next to the inversion with respect to p, i.e. $\mathscr{L}_p^{-1}$, one can see that the main problem in inverting equation (20) lies in the inversion of the typical term

$$
\begin{aligned}
r_n(t^+, p) &= \frac{1}{c_n p^{\frac{2}{3}}}\exp\left[-(c_n t^+)\, p^{\frac{2}{3}}\right] \\
&\text{for} \quad n \geqslant 1, \quad t^+ > 0 \quad \text{and} \quad \mathscr{R}(p) > 0. \qquad (21)
\end{aligned}
$$

This inversion is carried out using the standard inversion integral,

$$\mathscr{L}_p^{-1}\{r_n(t^+, p)\} = \frac{1}{2\pi i c_n} \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{1}{p^{\frac{2}{3}}} \exp\left[x^+ p - (c_n t^+)\, p^{\frac{2}{3}}\right] \mathrm{d}p, \; t^+ > 0, \; x^+ > 0. \tag{22}$$

The integration in equation (22) is taken along the Bromwich path in the p-plane, i.e. along $p = \gamma_0 + i\beta$. This integral is shown in the Appendix to reduce to the Airy function of square argument and the result of the inversion is

$$\mathscr{L}_p^{-1}\{r_n(t^+, p)\} = \frac{3^{\frac{2}{3}}}{c_n (x^+)^{\frac{1}{3}}} \exp\left[-\frac{2}{27}(c_n t^+/x^{+\frac{2}{3}})^3\right] \mathrm{Ai}\left[\frac{(c_n t^+/x^{+\frac{2}{3}})^2}{3^{\frac{4}{3}}}\right]. \tag{23}$$

The steady state solution of the wall heat flux $q_{w,\,ss}$, (i.e. the solution of equation (4) when the transient term $\partial T^+/\partial t^+$ vanishes), was given by Tribus and Klein [10], and has been also derived here independently as

$$q_{w,\,ss}(x^+)^{\frac{1}{3}}\, \nu/u_\infty\, T_w k \sigma^{\frac{1}{3}} = \frac{3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})}. \tag{24}$$

Hence the final result for the ratio of the time dependent heat flux at the wall to its steady state value ($q_w/q_{w,\,ss}$) may be written in the physical τ^+ plane where $\tau^+ = t^+/x^{\frac{2}{3}}$, with help of equations (15), (20), (23) and (24) as

$$q_w/q_{w,\,ss} = 1 + 3^{\frac{1}{3}}\,\Gamma(\tfrac{1}{3}) \sum_{n=1}^{\infty} \frac{1}{c_n} \exp\left[-\frac{2}{27}(c_n \tau^+)^3\right] \mathrm{Ai}\left[\left(\frac{c_n \tau^+}{3^{\frac{2}{3}}}\right)^2\right]. \tag{25}$$

This expression represents the exact analytical solution for the surface heat flux deduced from the partial differential equation (4) and the side conditions (5) and (6).

The first term on the right-hand side of equation (25) corresponds to the steady state part of the solution. The deviation from the steady state is accounted for by the infinite series. The number of terms required for the numerical evaluation of the series depends on how fast the exponential terms tend to zero. Therefore the smaller the values of $(c_n \tau^+)$ is the more terms in the series are needed. In the limit as $\tau^+ \to 0$ the form of equation (25) becomes unsuitable for numerical computations.

REARRANGEMENT AND NUMERICAL EVALUATION OF THE SOLUTION

Equation (25) can be expressed in a form more convenient for numerical computations if one uses the fact that the zeros of the Airy function are represented asymptotically by the simple expression [8]

$$\tilde{c}_n = \left[\frac{3\pi}{8}(4n - 1)\right]^{\frac{2}{3}}. \tag{26}$$

For $n \geqslant 7$ the values of c_n can be calculated to five significant figures from equation (26). Equation (25) can be re-written with help of this representation as

$$q_w/q_{w,\,ss} = 1 + A_{\mathrm{I}} + A_{\mathrm{II}} \tag{27}$$

where A_{I} is the sum of the first six terms and A_{II} is the remainder of the infinite series. The values of c_n in A_{II} are replaced by their asymptotic expression $\tilde{c}_n$. Using the trapezoidal formula for the approximate evaluation of integrals, [11] p. 215–220 and [1], A_{II} can be represented by

$$\begin{aligned} A_{\mathrm{II}} &\equiv N(\tau^+) + A_\alpha(\tau^+) + E(\tau^+) \\ &= 3^{\frac{1}{3}}\,\Gamma(\tfrac{1}{3}) \int_7^{\infty} \frac{1}{\tilde{c}_n} \exp\left[-\frac{2}{27}(\tilde{c}_n \tau^+)^3\right] \mathrm{Ai}\left[\left(\frac{\tilde{c}_n \tau^+}{3^{\frac{2}{3}}}\right)^2\right] \mathrm{d}n + \frac{3^{\frac{1}{3}}\,\Gamma(\frac{1}{3})}{2\alpha} \exp\left[-\frac{2}{27}(\alpha \tau^+)^3\right] \mathrm{Ai}\left[\left(\frac{\alpha}{3^{\frac{2}{3}}}\tau^+\right)^2\right] + E(\tau^+) \end{aligned} \tag{28}$$

where $\alpha = 9(3\pi/8)^{\frac{2}{3}}$.

$N(\tau^+)$ represents the first, $A_\alpha(\tau^+)$ the second and $E(\tau^+)$ the third term on the right-hand side of the equation (28). The error term $E(\tau^+)$ is

Table 1. The numerical evaluation of equation (35)

τ^+	A_1	A_2	A_3	A_4	A_5	A_6	N	A_α	E	$q_w/q_{w,ss}$
0·002	0·5867	0·3355	0·2485	0·2021	0·1727	0·1520	20·666	0·0683	0·0011	23·433
0·004	0·5867	0·3355	0·2485	0·2021	0·1726	0·1520	13·803	0·0683	0·0011	16·570
0·006	0·5867	0·3355	0·2485	0·2021	0·1726	0·1520	10·763	0·0683	0·0011	13·530
0·008	0·5866	0·3355	0·2484	0·2020	0·1726	0·1520	8·9503	0·0682	0·0011	11·717
0·010	0·5866	0·3355	0·2483	0·2020	0·1725	0·1518	7·7137	0·0682	0·0011	10·480
0·020	0·5865	0·3352	0·2479	0·2015	0·1719	0·1511	4·6473	0·0678	0·0011	7·4103
0·040	0·5858	0·3339	0·2462	0·1993	0·1693	0·1482	2·4898	0·0661	0·0010	5·2396
0·060	0·5846	0·3318	0·2432	0·1955	0·1647	0·1428	1·5513	0·0631	0·0009	4·2779
0·080	0·5829	0·3287	0·2388	0·1898	0·1579	0·1349	1·0122	0·0586	0·0009	3·7047
0·100	0·5807	0·3245	0·2328	0·1822	0·1488	0·1244	0·6665	0·0527	0·0009	3·3135
0·120	0·5780	0·3192	0·2252	0·1726	0·1374	0·1115	0·4344	0·0457	0·0009	3·0249
0·140	0·5746	0·3127	0·2161	0·1611	0·1240	0·0967	0·2764	0·0379	0·0009	2·8004
0·160	0·5706	0·3050	0·2053	0·1479	0·1090	0·0808	0·1700	0·0298	0·0009	2·6193
0·180	0·5660	0·2962	0·1930	0·1332	0·0932	0·0648	0·1001	0·0222	0·0007	2·4694
0·200	0·5608	0·2861	0·1795	0·1176	0·0771	0·0500	0·0559	0·0155	0·0003	2·3428
0·250	0·5448	0·2561	0·1413	0·0778	0·0408	0·0200	0·0045	0·0099	0·0001	2·0953
0·300	0·5250	0·2203	0·1013	0·0433	0·0164	0·0054	0·0011	0·0008	0·0000	1·9136
0·350	0·5001	0·1809	0·0650	0·0196	0·0047	0·0009	0·0001	0·0001	0	1·7714
0·400	0·4715	0·1408	0·0366	0·0070	0·0009	0·0001	0·0000	0·0000	0	1·6569
0·450	0·4392	0·1032	0·0178	0·0019	0·0001	0·0000	0	0	0	1·5622
0·500	0·4036	0·0707	0·0073	0·0004	0·0000	0	0	0	0	1·4820
0·550	0·3656	0·0450	0·0025	0·0001	0	0	0	0	0	1·4132
0·600	0·3259	0·0263	0·0007	0	0	0	0	0	0	1·3529
0·650	0·2856	0·0141	0·0002	0	0	0	0	0	0	1·2999
0·700	0·2457	0·0068	0·0000	0	0	0	0	0	0	1·2525
0·750	0·2073	0·0030	0	0	0	0	0	0	0	1·2103
0·800	0·1712	0·0012	0	0	0	0	0	0	0	1·1724
0·850	0·1383	0·0004	0	0	0	0	0	0	0	1·1387
0·900	0·1091	0·0001	0	0	0	0	0	0	0	1·1092
0·950	0·0839	0·0000	0	0	0	0	0	0	0	1·0839
1·000	0·0628	0	0	0	0	0	0	0	0	1·0628
1·100	0·0323	0	0	0	0	0	0	0	0	1·0323
1·200	0·0147	0	0	0	0	0	0	0	0	1·0147
1·300	0·0058	0	0	0	0	0	0	0	0	1·0058
1·400	0·0020	0	0	0	0	0	0	0	0	1·0020
1·500	0·0006	0	0	0	0	0	0	0	0	1·0006
1·600	0·0001	0	0	0	0	0	0	0	0	1·0001
1·700	0·0000	0	0	0	0	0	0	0	0	1·0000
1·800	0	0	0	0	0	0	0	0	0	1·0000

shown in [11] and [1] to be bounded by

$$E(\tau^+) = -\frac{1}{12}\frac{\partial}{\partial n}\left\{\frac{3^{\frac{1}{3}}\,\Gamma(\frac{1}{3})}{\tilde{c}_n}\exp\left[-\frac{2}{27}(\tilde{c}_n\tau^+)^3\right]\mathrm{Ai}\left[\left(\frac{\tilde{c}_n}{3^{\frac{1}{3}}}\tau^+\right)^2\right]\right\}\Bigg|_{n=7}. \qquad (29)$$

Carrying out the above differentiation results in three terms. The terms which are multiplied by $(\tau^+)^2$ and $(\tau^+)^3$ may be neglected. This is due to the fact that A_{II} and $E(\tau_+)$ are only of importance at short times as will be seen in Table 1. Thus equation (29) reduces to

$$E(\tau^+) = \frac{2\cdot 3^{\frac{1}{3}}\,\Gamma(\frac{1}{3})}{243\,\alpha}\exp\left[-\frac{2}{27}(\alpha\tau^+)^3\right]\mathrm{Ai}\left[\left(\frac{\alpha}{3^{\frac{1}{3}}}\tau^+\right)^2\right]. \qquad (30)$$

It is shown in [1] that the product of the exponential and the Airy functions, which have appeared repeatedly above, can be expressed in the infinite series form

$$3^{\frac{2}{3}} \exp\left[-\frac{2}{27}(c_n\tau^+)^3\right] \mathrm{Ai}\left[\left(\frac{c_n}{3^{\frac{2}{3}}}\tau^+\right)^2\right] = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\frac{2}{3}-\frac{2}{3}m)}(c_n\tau^+)^m. \qquad (31)$$

In order to evaluate the integral $N(\tau^+)$ in equation (28), one first uses the substitution

$$\frac{2}{27}\left[\left(\frac{3\pi}{8}\right)^{\frac{2}{3}}(4n-1)^{\frac{2}{3}}\tau^+\right]^3 = z. \qquad (32)$$

With this

$$N(\tau^+) = \frac{3^{\frac{1}{3}}\,\Gamma(\frac{1}{3})}{2^{\frac{1}{6}}\,.\,3^{\frac{1}{2}}\pi} \times \frac{1}{\sqrt{\tau^+}}\left\{\int_0^{\infty} z^{-\frac{5}{6}}\exp(-z)\,\mathrm{Ai}\left[\left(\frac{3z}{2}\right)^{\frac{2}{3}}\right]\mathrm{d}z - \int_0^{\frac{2}{27}(\alpha\tau^+)^3} z^{-\frac{5}{6}}\exp(-z)\,\mathrm{Ai}\left[\left(\frac{3z}{2}\right)^{\frac{2}{3}}\right]\mathrm{d}z\right\}. \qquad (33)$$

The first integral can be readily evaluated if Ai is expressed in terms of $K_{\frac{1}{3}}$. If Ai is expressed in terms of $I_{-\frac{1}{3}}$ and $I_{\frac{1}{3}}$ in the second integral one can rewrite it as an infinite series of the modified

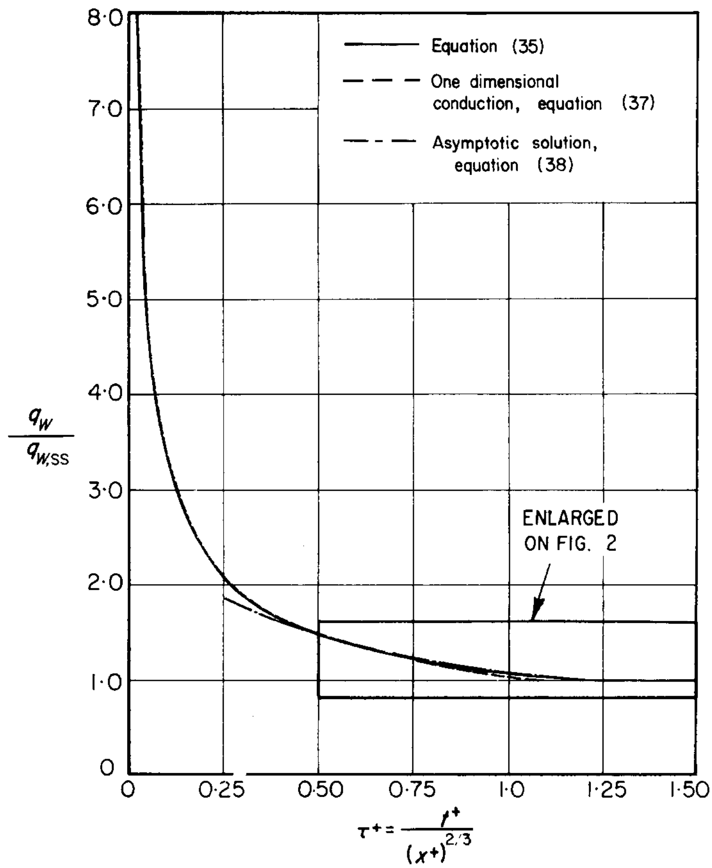

FIG. 1. Heat-transfer response to a step in wall temperature.

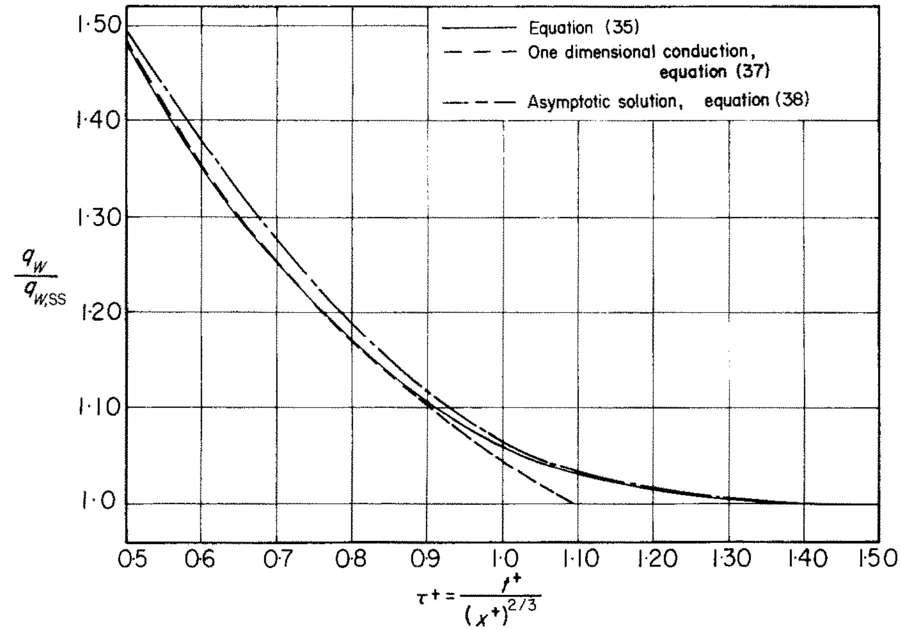

FIG. 2. Heat-transfer response to a step in wall temperature.

Bessel functions of the first kind, see [12] pp. 100 and 107. However the final result is obtained in a more compact form if the relationship (31) is substituted in the second integral and the integration is carried out. One obtains then

$$N(\tau^+) = \frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{3}}\sqrt{\pi}}\frac{1}{\sqrt{\tau^+}} - \frac{2 \, . \, 3^{\frac{1}{3}}\,\Gamma(\frac{4}{3})\sqrt{\alpha}}{\pi} \times \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)\,m!\,\Gamma(\frac{2}{3}-\frac{2}{3}m)}(\alpha\tau^+)^m. \quad (34)$$

The ratio $q_w/q_{w,ss}$ can now be expressed in a form which is more suitable for numerical computations, for small values of τ^+, than the form given in equation (25) i.e.

$$q_w/q_{w,ss} = 1 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + N + A_\alpha + E. \quad (35)$$

The terms $A_{1,2,\ldots 6}$ are the first six terms of the infinite series in equation (25). N is defined by equation (34), A_α is defined in equation (28) and E is given by equation (30). The individual terms in equation (35) have been evaluated numerically on an IBM 7094 computer and the results are presented in Table 1 together with the ratio $q_w/q_{w,ss}$ which is also shown in Figs. 1 and 2.

DISCUSSION

The present solution for the surface heat flux has been obtained by a double Laplace transformation. Equation (25) represents the exact analytical solution while equation (35) expresses the result in a convenient computational form within the error term E which is shown from Table 1 to be extremely small.

The terms N, A_α and E correspond to the remainder of the series for $n \geqslant 7$ (i.e. A_{II} of equation 27). Their effects are shown in Table 1 to be of importance only at short times. The terms $A_{1,2,\ldots 6}$ (i.e. A_I) are the transient contributions at large times. Each of these terms, starting with A_6, tends to zero as τ^+ gets larger (see Table 1). The value $q_w/q_{w,ss} = 1$ is attained to five significant figures at $\tau^+ = 1{\cdot}7$. It may be noted that the Prandtl number does not appear as a separate parameter in the present problem. The wall heat flux q_w is shown by equations (24) and (25) to be proportional to $\sigma^{\frac{1}{3}}$ at all times.

In order to examine the behavior of the solution for short times, it is expanded in powers of τ^+. The result is, [1],

$$q_w/q_{w,\,ss} = \frac{1\cdot 0480}{\sqrt{\tau^+}} - 0\cdot 1376 + 0\,.(\tau^+) + 6\cdot 9177\,(\tau^+)^2 + 10\cdot 029\,(\tau^+)^3 - 0\,.(\tau^+)^4 \ldots \quad (36)$$

Equation (36) shows that the solution is expressed, from the second term on, in integer powers of τ^+. The first term on the right-hand side of equation (36) appears as a result of the use of the asymptotic form of the zeros of the Airy function. It may be of interest to note that analyses such as the one by Cess [2] and Riley [4] have proceeded by assuming *a priori* a series form of the wall heat flux as a function of τ^+. This is not required in the present analysis.

As $\tau^+ \to 0$, equation (36) reduces to

$$q_w/q_{w,\,ss} \sim \frac{1\cdot 0480}{\sqrt{\tau^+}}. \quad (37)$$

The above result is identical to the response of the surface heat flux of a semi-infinite solid when its boundary is suddenly brought to a temperature T_w for $t > 0$. The same observation was noted in references [2–5] and many others.

For large times the only term which controls the approach to the steady state, as seen from Table 1, is A_1. If one uses the asymptotic expansion of the Airy function, as $\tau^+ \to \infty$ equations (35) or (25) reduce, in the final decay to the steady state, to

$$q_w/q_{w,\,ss} \sim 1 + \frac{3^{\frac{4}{3}}\,\Gamma(\frac{1}{3})}{2\sqrt{\pi}\,c_1}\,\frac{1}{(c_1\tau^+)^{\frac{1}{4}}} \exp\left[-\frac{4}{27}(c_1\tau^+)^3\right]. \quad (38)$$

Equation (38) shows that the steady state is approached exponentially. A similar conclusion was reached by Riley [4]. Equations (37) and (38), valid for small and for large times respectively, are shown also in Figs. 1 and 2.

At last we outline the method for the determination of the surface temperature due to a time step q_0 in the surface heat flux. In this case a dimensionless temperature is defined by $T^+ = T\,k\,u_\infty\,\sigma^{\frac{1}{3}}/q_0\nu$ and together with the variables introduced in equation (3) there result once more the equations (4) and (5) while the first boundary condition in equation (6) is replaced by

$$-\frac{\partial T^+}{\partial y^+}(t^+, x^+, 0) = 1, \quad t^+ > 0, \quad x^+ > 0. \quad (39)$$

Following the steps in the Analysis Section one derives in place of equation (14) the following expression for the transformed dimensionless surface temperature in the (s, p) domain.

$$\varphi(s, p, s) = -\frac{\mathrm{Ai}(s/p^{\frac{2}{3}})}{s p^{\frac{1}{3}}\,\mathrm{Ai}'(s/p^{\frac{2}{3}})}. \quad (40)$$

The inversion with respect to s yields then

$$\mathscr{L}_s^{-1}\{\varphi(s, p, s)\} = \frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{3}}\,\Gamma(\frac{2}{3})}\,\frac{1}{p^{\frac{1}{3}}} - \sum_{n=1}^{\infty} \frac{1}{(c_n')^2 p^{\frac{1}{3}}} \exp\left[-(c_n' t^+)p^{\frac{2}{3}}\right], \quad t^+ > 0, \quad \mathscr{R}(p) > 0 \quad (41)$$

where the c_n' are the zeros of Ai′ and are tabulated in [8].

If one compares the infinite series part in equation (41), i.e. $F_{1\,\infty}(t^+, p; c_n')$ with the corresponding term in equation (20), i.e. $F_\infty(t^+, p; c_n)$ one deduces the following relationship between the two functions

$$F_{1\infty}(t^+, p; c_n') = \int_{t^+}^{\infty} F_\infty(t^+, p; c_n')\,\mathrm{d}t^+. \quad (42)$$

On interchanging the order of integration the inversion with respect to p yields then

$$\mathscr{L}_p^{-1}\{F_{1\infty}(t^+, p; c_n')\} = \int_{t^+}^{\infty} \mathscr{L}_p^{-1}\{F_\infty(t^+, p; c_n')\}\,\mathrm{d}t^+. \quad (43)$$

However, the inversion of the integrand on the right-hand side has already been carried out in the Analysis Section and one has only to replace the value of c_n by c_n' in this result according to equation (43). On inverting also the first term in equation (41) one obtains finally the quotient of the transient surface temperature to its steady state value

$$\frac{T_w}{T_{w,\,ss}} = 1 - \Gamma(\tfrac{2}{3}) \sum_{n=1}^{\infty} \frac{1}{c_n'} \int_{\tau^+}^{\infty} \exp\left[-\frac{2}{27}(c_n'\tau^+)^3\right] \mathrm{Ai}[(c_n'\tau^+/3^{\frac{4}{3}})^2]\, d\tau^+ \tag{44}$$

where

$$\frac{T_{w,\,ss}\, k\, u_\infty\, \sigma^{\frac{1}{3}}}{q_0 \nu} = \frac{3^{\frac{2}{3}}}{\Gamma(\frac{2}{3})}(x^+)^{\frac{1}{3}}.$$

The integral in equation (44) can be readily computed [12]. Making use of the asymptotic representation for the zeros of Ai′, i.e. c_n' one may rearrange equation (44) in a form comparable to equation (35) and obtain a useful form for numerical evaluation.

CONCLUSION

Exact analytical solutions have been obtained for the classical time-dependent Lévêque problem described by equation (4). These solutions may serve as a reliable reference to check techniques developed for the treatment of more complicated problems. The initial behavior of the solution reveals that for small times the heat-transfer process is controlled by one dimensional diffusion. For large times the solution demonstrates that the steady state heat-transfer process is approached in an exponential manner.

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APPENDIX

Evaluation of the inversion integral

Consider the two Argand planes $p = \gamma + i\beta$, $\eta = \lambda + i\omega$ and let

$$\eta = p^{\frac{1}{3}} \tag{A-1}$$

in equation (22) with the result

$$\mathscr{L}_p^{-1}\{r_n(t^+, p)\} = \frac{3}{2\pi i c_n} \int_{D_1} \exp(x^+\eta^3 - t^+\eta^2)\, d\eta \tag{A-2}$$

where D_1 is the mapped Bromwich path (i.e. $\gamma = \gamma_0$) in the η-plane. The principal branch of $p^{\frac{1}{3}}$ maps the entire p-plane into the sector bounded by the dashed lines in Fig. 3. If equation (A-1) is expressed in terms of real and imaginary parts there follows that the equation of the path D_1 is given by

$$\lambda^3 - 3\lambda\omega^2 = \gamma_0\,. \tag{A-3}$$

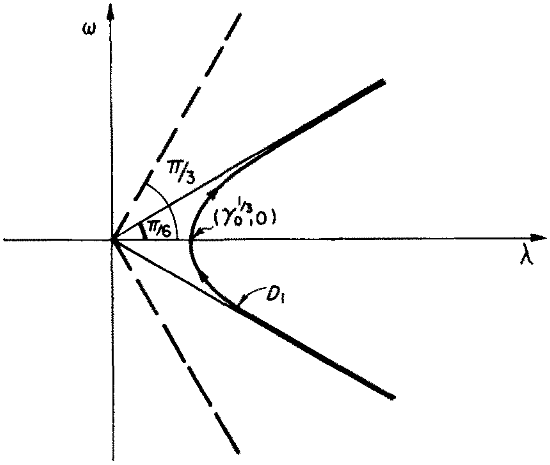

FIG. 3. The η-plane.

The path D_1 is shown in Fig. 3, together with its asymptotes which make an angle $\pm\pi/6$ with positive real axis.

Now, choose $\gamma_0 = (c_n t^+/3x^+)^3$ and use the mapping function

$$\eta = \left(\frac{c_n t^+}{3x^+}\right)(1 - \bar{\eta}), \qquad \bar{\eta} = \bar{\lambda} + i\,\bar{\omega} \qquad \text{(A-4)}$$

in equation (A-2) to get

$$\mathscr{L}_p^{-1}\{r_n(t^+, p)\} = \frac{3^{\frac{2}{3}}}{c_n(x^+)^{\frac{1}{3}}} \exp\left[-\frac{2}{27}(c_n\tau^+)^3\right] \left\{\frac{c_n\tau^+/3^{\frac{2}{3}}}{2\pi i}\int_{D_2} \exp\left[\left(\frac{c_n\tau^+}{3^{\frac{2}{3}}}\right)^3 (\bar{\eta} - \tfrac{1}{3}\bar{\eta}^3)\right] d\bar{\eta}\right\}. \qquad \text{(A-5)}$$

The equation of the path D_2 may be easily obtained from equations (A-3) and (A-4) as

$$(1 - \bar{\lambda})^3 - 3\bar{\omega}^2(1 - \bar{\lambda}) = 1. \qquad \text{(A-6)}$$

A straightforward aplication of Cauchy's Theorem, shows that the path D_2 can be replaced by a path D_3 running from $\infty \exp(-2\pi i/3)$ to 0 and then to $\infty \exp(+2\pi i/3)$ as shown in Fig. 4 together with the path D_2. The term in brackets

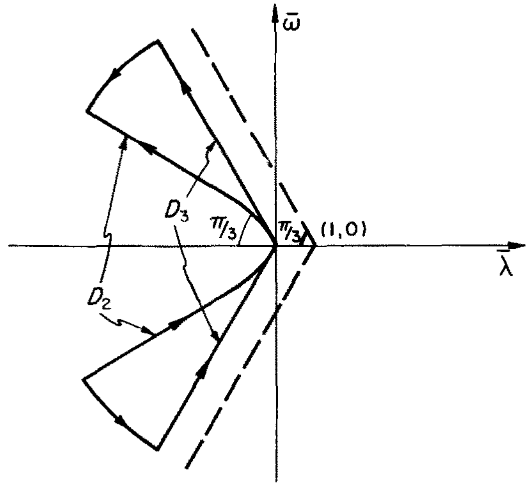

FIG. 4. The $\bar{\eta}$-plane.

in equation (A-5) with the integral taken along the path D_3, is shown in reference [13] to be an integral representation of the Airy function of square argument. Hence equation (23) follows.

Résumé—Le problème classique de Lévêque en régime permanent est généralisé pour tenir compte de la dépendance du temps. On présente des solutions analytiques exactes pour le flux de chaleur pariétal dû à une variation en échelon dans le temps de la température pariétale et pour la température pariétale due à une variation, en échelon dans le temps du flux de chaleur pariétal. Le comportement initial et final est étudié théoriquement et pendant toute l'évolution temporelle, l'on y a ajouté une évaluation numérique, dans le cas d'un échelon de température pariétale.

Zusammenfassung—Das klassische stationäre Lèvêque-Problem wird verallgemeinert um Zeitabhängigkeiten zu berücksichtigen. Exakte analytische Lösungen werden angegeben für den Wärmefluss an der Oberfläche infolge eines Zeitschrittes in der Wandtemperatur und für die Wandtemperatur bei einem Zeitschritt im Wärmestrom. Das Anfangs- und Endverhalten der Lösung wird analytisch untersucht und durch eine numerische Auswertung für die gesamte Zeitspanne ergänzt für den Fall einer schrittweisen Änderung der Wandtemperatur.

Аннотация—Классическая стационарная задача Левека обобщена для случая с учетом временной зависимости. Приведены аналитические решения для двух случаев: (1) тепловой поток стенки вызван временным скачком температуры на ней; (2) температура поверхности обусловлена временным скачком теплового потока на ней. Начальное и конечное временное поведение решения изучаются аналитически, и численно определяется полный период времени для случая скачкообразного изменения температуры стенки.